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HARMONIC FORMS.—THEIR CONSTRUCTION REDUCED TO A SCIENCE.

By BERNARD B. SMYTH, Topeka.

PART I. PLANIMETRIC FORMS.

1. It is possible to so arrange arithmetical series, either regular or varying, in any regular form that, upon addition of the numbers in any stated direction, equal or proportional sums will be obtained.

2. Trigonal, square, and other figurate series may be arranged in circles, hexagons, and other forms so that addition in various directions will give equal or proportional sums.

3. Geometric series may be arranged, like arithmetical series, so that continued products in any direction will equal continued products of an equal number of factors in any other direction.

In arithmetical series, among the forms that have been successfully tried are squares, rectangles, triangles, hexagons, octagons, four-, five-, six-, and eight-pointed stars, circles, ellipses, cubes, parallelopipeds, prisms, pyramids, cylinders, ellipsoids, and spheres.

The principles of construction, so far as relate to plane forms, are here shown:

SECTION I. SQUARES.

Square of Two.

The smallest square that may be attempted is one of four places. It gives equal sums only twice—either vertically, horizontally, or diagonally, according to arrangement, thus:

1	2
4	3

1	4
2	3

1	2
3	4

As to differences, there are three differences of one unit each between two contiguous numbers, namely: between 1 and 2, between 2 and 3, and between 3 and 4; two differences of two units each, namely, between 1 and 3, and between 2 and 4; and one difference of three units, namely, between 1 and 4. These differences perform an important part in the formation of higher squares, as of 6, 10, etc.

As to laying the four numbers, there are twenty-four different ways of laying them, thus: When 1 is placed in the upper left-hand corner, there is a choice of *three* squares in which 2 may be laid; 1 and 2 being laid, there is a choice of two squares in which 3 may be placed, and then there is but *one* place for 4. Multiplying these (3, 2, and 1) together gives 6 as the possible arrangements when 1 is placed in a certain corner; but there are four corners: Hence there are twenty-four different arrangements by which these four numbers may be laid.

This square is not harmonic, and is introduced here because it is a factor in the construction of harmonic squares of six, ten, fourteen, etc., places on a side

Square of Three.

The smallest square that can be constructed that will sum up equally in all its lines, columns, and diagonals, is one of nine places, thus:

(1)

8	1	6
3	5	7
4	9	2

(2)

6	1	8
7	5	3
2	9	4

(3)

4	9	2
3	5	7
8	1	6

(4)

2	9	4
7	5	3
6	1	8

(5)

8	3	4
1	5	9
6	7	2

(6)

6	7	2
1	5	9
8	3	4

(7)

4	3	8
9	5	1
2	7	6

(8)

2	7	6
9	5	1
4	3	8

This, though apparently eight different squares, is really only one square presented in eight different aspects by transposition.

These schemes show the mode of arrangement:

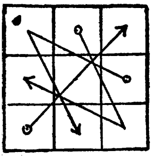


Fig. 1.

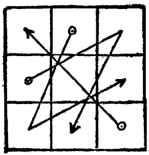


Fig. 2.

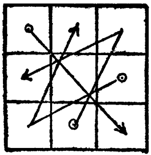


Fig. 3.

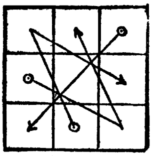


Fig. 4.

This scheme presupposes that the nine numbers are arranged in three sets; and that the initial and final sets are arranged along the broken arrows, and the medial set along the diagonal straight arrow. Figure 1, in the scheme above, answers for squares 1 and 8; fig. 2 answers for squares 2 and 6; fig. 3 for squares 3 and 7; and fig. 4 for squares 4 and 5. It matters not on which side a beginning be made, nor whether the three numbers of a set increase in the direction of an arrow point or the reverse (though in the scheme above, the numbers are supposed to increase toward the arrow points); but whatever arrangement be made for one set must be observed for all sets in the same square.

That no more than eight sums of 15 from three numbers each can be obtained from these nine numbers may be determined by arranging the numbers in a circle and connecting by lines any three numbers that will add 15, thus:

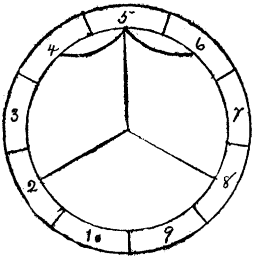


Fig. 5.

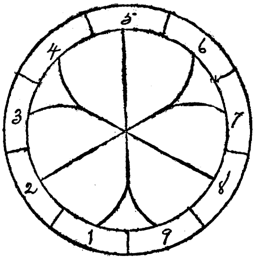


Fig. 6.

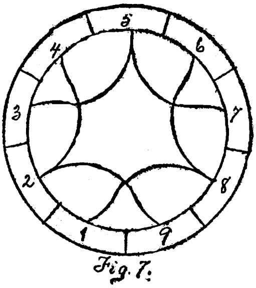


Fig. 7.

On arranging the numbers in a circle, the first thing most apparent is that the circle may be divided into three equal parts by the equidistant numbers, 2, 5, and 8, the sum of which equals 15; also, that the sum of the three adjacent numbers at the top of the circle equals 15. (See fig. 5.) If we try the other equidistant numbers we find they will not make 15. We learn then, that the numbers are divided into three sets of three numbers each, and that both sets and numbers have an initial, a medial, and a final.

Next, we find that each of the medial numbers may be connected with a pair directly opposite—that is: with the initial of the preceding set, and the final of the following set. (See fig. 6.)

And, finally, we find that each medial number may be connected with the final of the preceding set and the initial of the following set. (See fig. 7.)

This makes eight possible sums of 15. The circle complete, which is shown above as three separate circles in order to exhibit the successive steps in the possible additions, is here shown in its entirety:

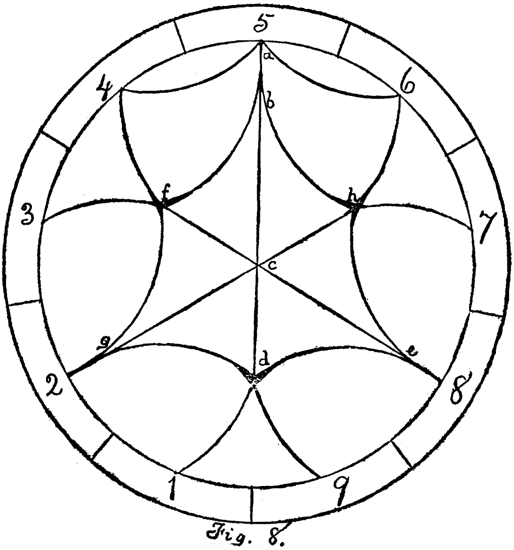


Fig. 8.

Each of the forks (a, b, c, etc.) in the above circle connects three numbers in the margin either by a straight or a curved line. The sum of these three numbers is in all cases 15. There are eight of these forks. It will readily be seen that to connect the numbers by threes in other ways than as shown, and still have the three numbers add 15, is impossible.

It may be noticed as a peculiarity that the three medial numbers (4, 5, 6,) form one of the diagonals in the square; and the three numbers of the medial set (2, 5, 8,) form the other diagonal. The lines and columns have each an initial, a medial, and a final.

Again, in an apparently simpler method, it may be seen by simply arranging the numbers in a square, in regular order, thus:

(9)

1	2	3
4	5	6
7	8	9

Here we see that the two diagonals, the central line, and the central column, add each 15. It may also be seen that each of the four corners, with the two numbers diagonally across the opposite corner, as indicated in the scheme above (figs. 1 and 2) separately foot up 15. For example: $1 + 6 + 8 = 15$, $3 + 4 + 8 = 15$, etc. This gives the clue for the scheme (figs. 1 to 4).

Square of Four.

Squares of 16 places have long been considered very difficult, and if by chance a "magic" square were constructed it was heralded as a great accomplishment. Yet, when the principles of their construction are fully understood, the making of these squares by the hundred or the thousand becomes a comparatively easy matter.

I will first construct a square, and we will then proceed to analyze it and study the principles. Take this as a model:

No. 1.

1	14	12	7
15	4	6	9
8	11	13	2
10	5	3	16

Is this square harmonic? Let us see:

Equal sums are obtained by adding together four numbers in each of the following regular ways:

1st.— By lines —

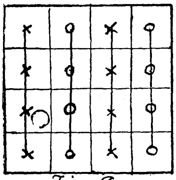


Fig. 9.

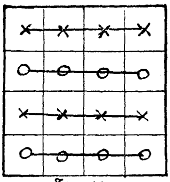


Fig. 10.

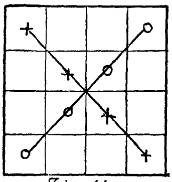


Fig. 11.

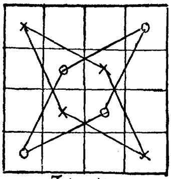


Fig. 12.

	Ways.
Each of the columns (fig. 9)	4
Each of the lines (fig. 10)	4
Each of the diagonals (fig. 11)	2

2d.— By squares —

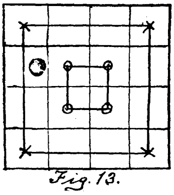


Fig. 13.

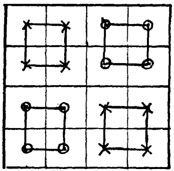


Fig. 14.

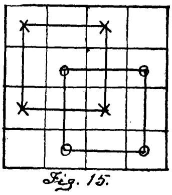


Fig. 15.

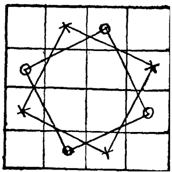


Fig. 16.

The four corners and the four centrals (fig. 13)	2
The four of each quarter (fig. 14)	4
Corresponding corners of quarters (fig. 15)	4
Knight-spring numbers in the margin (fig. 16)	2

3d.— By rectangles —

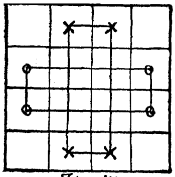


Fig. 17.

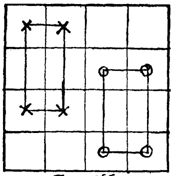


Fig. 18.

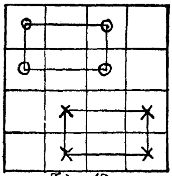


Fig. 19.

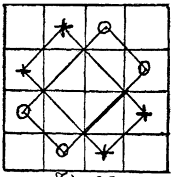


Fig. 20.

Extremes of medial columns and lines (fig. 17)	2
Alternate linear pairs (fig. 18)	4
Alternate columnar pairs (fig. 19)	4
Transverse diagonals of opposite quarters (fig. 20)	2

4th.—*By rhombs and rhomboids*—

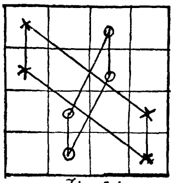


Fig. 21.

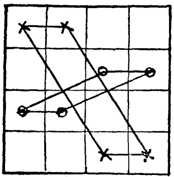


Fig. 22

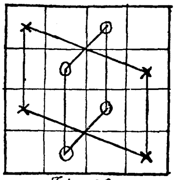


Fig. 23.

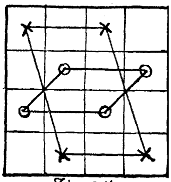


Fig. 24.

Extremes of one diagonal and means of the other (fig. 12)	2
Opposite columnar pairs (fig. 21)	4
Opposite linear pairs (fig. 22)	4
Alternating numbers in opposite columns (fig. 23)	4
Alternating numbers in opposite lines (fig. 24)	4
Total ways of adding 34	52

Here, then, are 52 *regular* ways of adding equal sums (in this case 34). There are many other ways, more or less irregular, of obtaining the same result, as 12, 15, 4, 3, etc.; but the only notice taken of them is that, whatever order be taken in obtaining the sum of 34, the sum can be obtained from four numbers in corresponding positions diametrically opposite.

We may readily concede, then, that this square is perfectly harmonic in all its parts. All parts of the square are evenly balanced and corresponding with every other part.

TRANSPOSITIONS.

Taking this square as a primary, 15 other squares may be derived from this by regular transpositions. All of these *derived* squares will have the same attributes as the primary square; will add equal sums in all the different ways, and in the same order, as the primary square; will, in short, be as perfectly harmonic.

In order to express the different transpositions, the most expressive words will be used; and, where the language does not furnish a properly expressive word, a word suiting the purpose will be invented from an arbitrary syllable.

Square No. 1 is here reproduced for the purpose of comparison with the 15 derived squares.

No. 1.
Primary.

1	14	12	7
15	4	6	9
8	11	13	2
10	5	3	16

No. 2.
By Troversion.

14	1	7	12
4	15	9	6
11	8	2	13
5	10	16	3

No. 3.
By Diversion.

12	7	1	14
6	9	15	4
13	2	8	11
3	16	10	5

No. 4.
By Reversion.

7	12	14	1
9	6	4	15
2	13	11	8
16	3	5	10

No. 2 is obtained by transposing adjacent columns, as the second with the first and the fourth with the third. To express this change the syllable *tro*, with the meaning of *around*, will be used. Hence, this will be called transposing by *troversion*. This puts the number 14 in the upper-left-hand corner—the key corner.

No. 3 is formed by *di*viding No. 1 *ver*tically and placing the last half before the first half. Mnemonically, we take the first syllables of the two expressive words, as above, and call this transposing by *diversion*. This puts 12 in the initial or key corner.

No. 4 is obtained by *reversing* the lines and placing 7 in the key corner. It is called transposing by *reversion*.

No. 5.
By Conversion.

15	4	6	9
1	14	12	7
10	5	3	16
8	11	13	2

No. 6.
By Controversion.

4	15	9	6
14	1	7	12
5	10	16	3
11	8	2	13

No. 7.
By Condiversion.

6	9	15	4
12	7	1	14
3	16	10	5
13	2	8	11

No. 8.
By Reconversion.

9	6	4	15
7	12	14	1
16	3	5	10
2	13	11	8

No. 5 is obtained by transposing adjacent lines of No. 1, the second with the first and the fourth with the third; as the columns were transposed to form No. 2. For this, the syllable *con*, with the sense of *over*, will be used. This, then, is transposing by *conversion*, and places 15 in the key corner.

No. 6 is derived from No. 1 by *converting* the *troverted* square No. 2 or by *troverting* the *converted* square No. 5. This is transposing by *controversion*, and puts 4 in the upper left-hand corner.

No. 7 is obtained from No. 1 by *converting* the *diverted* square No. 3, or by *diverting* the *converted* square No. 5. This is called transposing by *condiversion*, and brings 6 into the upper left-hand corner.

No. 8 is obtained by reversing the lines of No. 5 or by converting No. 4. This is called *reconversion* and brings 9 into the key corner.

No. 9.
By Subversion.

8	11	13	2
10	5	3	16
1	14	12	7
15	4	6	9

No. 10
By Subtroversion.

11	8	2	13
5	10	16	3
14	1	7	12
4	15	9	6

No. 11
By Subdiversion.

13	2	8	11
3	16	10	5
12	7	1	14
6	9	15	4

No. 12.
By Resubversion.

2	13	11	8
16	3	5	10
7	12	14	1
9	6	4	15

No. 9 is obtained by superimposing the lower half of No. 1 above the upper half. This is called *subversion.* It places 8 in the key corner.

No. 10 is obtained by *troverting* No. 9, or by *subverting* No. 2. It is called transposing by *subtroversion.* It brings 11 into the key corner.

No. 11 comes by *subverting* No. 3 or by *diverting* No. 9. It is called transposing by *subdiversion.* It brings 13 into the key corner.

No. 12 is made by *reversing* No. 9. This is known as *resubversion.* In this square, 2 comes to the key corner.

No. 13.
By Inversion.

10	5	3	16
8	11	13	2
15	4	6	9
1	14	12	7

No. 14.
By Introversion.

5	10	16	3
11	8	2	13
4	15	9	6
14	1	7	12

No. 15.
By Indiversion.

3	16	10	5
13	2	8	11
6	9	15	4
12	7	1	14

No. 16.
By Reinversion.

16	3	5	10
2	13	11	8
9	6	4	15
7	12	14	1

No. 13 is obtained by *inverting* No. 1; No. 14 by *troverting* No. 13; No. 15 by *diverting* No. 13; and No. 16 by *reversing* No. 13. These squares have respectively the numbers 10, 5, 3, and 16 in the key corners.

Each of these fifteen derived squares adds 34 in all the regular ways enumerated for No. 1.

In the sixteen transposed squares above, each of the successive numbers is brought in succession into the key position, and in the exact order in which the same numbers occur in square No. 1. Now, by placing the respective syllables that express the mode of transposition in the square, letting each occupy the position occupied by its key number in square No. 1, we have a sort of *key* to the name of a square after it is formed, and to the number that should occupy the upper left-hand corner when the square is formed:

KEY.

1	tro-	di-	re-
con-	contro-	condi-	recon-
sub-	subtro-	subdi-	resub-
in-	intro-	indi-	rein-

} -version.

Plan of Construction.

The sixteen numbers of the square are considered as consisting of four quaterniads or "sets" of four consecutive numbers each.

Additional Definitions.

By the term "couplet" is meant the two consecutive numbers, an odd and an even, constituting the anterior or the posterior half of a quaterniad, as 1 and 2, 5 and 6, 11 and 12, etc. Numbers falling *adjacently* in the square, as 1 and 15 or 2 and 11 in the squares above, are *columnar pairs* or *linear pairs,* according to whether they fall in the same column or same line. A pair in one square is a pair in all sixteen. Indeed that is the key to retaining the harmony of a square throughout all transpositions — the four numbers of any quarter are never separated; but, however the numbers of one quarter are shifted about, the numbers of all the other quarters are shifted about in a similar manner.

Let it be fully understood that, whether applying to lines, columns, pairs, single numbers, couplets, or quaterniads, the first and second, also the third and fourth, are *adjacent;* the second and third (though contiguous), also the first and fourth (or those which are complementary to each other in any other positions), are *opposite;* the first and third, also the second and fourth (or any occurring with an intervening one), are *alternate.* Adjacent quarters are those on the same side of a square, whether vertical or horizontal; opposite quarters are those diametrically opposite.

General Rules.

The arrangement of numbers in a square is largely optional; yet the following restrictions must be observed:

1. Two odd numbers, and no more, must be placed in each line, each column, and each central diagonal.
2. The position of the odd and even numbers in the square must be determined beforehand according to a certain *scheme* to be agreed upon.
3. The number 1 must always be placed in the upper left-hand corner.
4. The number 2 will be placed at the opposite end of a certain "coupling line" to be seen in the predetermined scheme.
5. The adjacent couplet (3 and 4) may have an arrangement the reverse or crosswise of, or parallel to, the first couplet, in other columns and lines. (The best arrangement is when the four numbers of a quaterniad are so placed that no two occupy the same line, column, or diagonal.)
6. Regarding the quaterniad already laid as having a "dexter" (or sinister) arrangement, the adjacent quaterniad (5, 6, 7, 8) must have a "sinister" (or dexter) arrangement, the couplets in reversed order, in the opposite half of the square (whether horizontal, vertical, exterior or interior) as indicated by the coupling lines of the scheme.
7. The opposite quaterniads must have a similar arrangement, the couplets in reversed order (though some other order will answer to a degree), in the opposite quarters of the square.
8. Thus arranged, numbers diametrically opposite will add together 17. This, though best, is not strictly essential.

Now we will consider the scheme by which our square "No. 1" is constructed:

Scheme I (47 of Schedule).

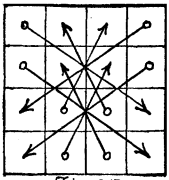


Fig 25.

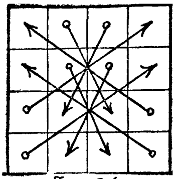


Fig. 26.

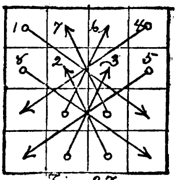


Fig. 27.

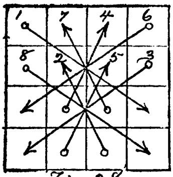


Fig. 28.

The scheme has two forms — a regular (fig. 25) and an inverted (fig. 26). The original or primary square is constructed according to fig. 25. Seven of the derived squares will be like fig. 25; eight of them like fig. 26.

In the above figures (and in all other schemes) the lines connect couplets of adjacent numbers of the series; the second couplet of a quaterniad is to be in the opposite half of the square. Thus, in the scheme above and our square No. 1, the first couplet occupies the upper alternate and exterior columnar half; the second couplet occupies the lower alternate and interior columnar half. The next quaterniad is just the reverse, the first couplet occupying the lower alternate interior half; the second couplet, the upper alternate exterior half.

Now, after having constructed the scheme — which is the first thing necessary to do in forming any harmonic square — the next step to take is to determine the *arrangement* of the numbers in the square. To do this, the coupling lines are to be numbered from 1 to 8 at their uppermost end. These numbers will then be in two rows or lines, with number 1 in the upper left-hand corner.

Numerous arrangements of these eight numbers may be made; and when made under certain laws a harmonic square may always be produced. Such arrangements may be divided into three series.

SERIES 1.—HARMONIC SQUARES.

To produce perfectly harmonic squares, so arrange these eight numbers with 1 in the upper left-hand corner that they shall add 18 in each line and 9 in each column. These may be called *trial arrangements.*

Under these restrictions, only six arrangements can be made, which, with their corresponding primary squares, are here produced:

1st Arr.

1 7 6 4
8 2 3 5

No. 1.

1	14	12	7
15	4	6	9
8	11	13	2
10	5	3	16

2d Arr.

1 7 4 6
8 2 5 3

No. 17.

1	14	8	11
15	4	10	5
12	7	13	2
6	9	3	16

3d Arr.

1 6 7 4
8 3 2 5

No. 18.

1	12	14	7
15	6	4	9
8	13	11	2
10	3	5	16

The first (see fig. 27) is our square No. 1, from which Nos. 2 to 16 were produced.

The second arrangement (fig. 28) gives square No. 17, from which 15 other squares are derived in the same manner as the first 16 from No. 1. These are all different from the first 16.

From the third arrangement (fig. 29) another set of 16 squares can be constructed by transposition the same as the first 16 were from the first arrangement.

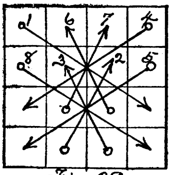


Fig. 29.

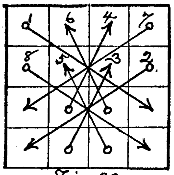


Fig. 30.

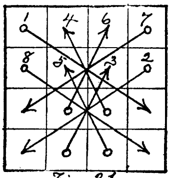


Fig. 31.

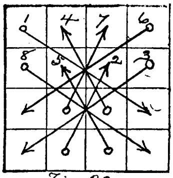


Fig. 32.

4th Arr.

$\left\{\begin{matrix}1 & 6 & 4 & 7\\ 8 & 3 & 5 & 2\end{matrix}\right\}$

No. 19.

1	12	8	13
15	6	10	3
14	7	11	2
4	9	5	16

5th Arr.

$\left\{\begin{matrix}1 & 4 & 6 & 7\\ 8 & 5 & 3 & 2\end{matrix}\right\}$

No. 20.

1	8	12	13
15	10	6	3
14	11	7	2
4	5	9	16

6th Arr.

$\left\{\begin{matrix}1 & 4 & 7 & 6\\ 8 & 5 & 2 & 3\end{matrix}\right\}$

No. 21.

1	8	14	11
15	10	4	5
12	13	7	2
6	3	9	16

From the 4th, 5th and 6th arrangements (figs. 30, 31, and 32, respectively,) 48 more squares may be constructed, making in all so far 96 harmonic squares.

SERIES 2.—SUBHARMONIC SQUARES.

In this series the squares add equally, vertically, horizontally, diagonally, and many other ways, but not quite so many as in the first series; yet they will undoubtedly give perfect satisfaction.

In this series of trial arrangements, the lines add 18 each; the four extremes add 18; the four means 18; the first four 18; and the last four 18. The resulting squares will be very harmonic, but rather less perfectly so than in series 1.

First Group.

7th Arr.

$\left\{\begin{matrix}1 & 8 & 5 & 4\\ 7 & 2 & 3 & 6\end{matrix}\right\}$

No. 22.

1	16	10	7
13	4	6	11
8	9	15	2
12	5	3	14

8th Arr.

$\left\{\begin{matrix}1 & 8 & 5 & 4\\ 6 & 3 & 2 & 7\end{matrix}\right\}$

No. 23.

1	16	10	7
11	6	4	13
8	9	15	2
14	3	5	12

9th Arr.

$\left\{\begin{matrix}1 & 8 & 3 & 6\\ 7 & 2 & 5 & 4\end{matrix}\right\}$

No. 24.

1	16	6	11
13	4	10	7
12	5	15	2
8	9	3	14

10th Arr.

$\left\{\begin{matrix}1 & 8 & 3 & 6\\ 4 & 5 & 2 & 7\end{matrix}\right\}$

No. 25.

1	16	6	11
7	10	4	13
12	5	15	2
14	3	9	8

11th Arr.

$\left\{\begin{matrix}1 & 8 & 2 & 7\\ 6 & 3 & 5 & 4\end{matrix}\right\}$

No. 26.

1	16	4	13
11	6	10	7
14	3	15	2
8	9	5	12

12th Arr.

$\left\{\begin{matrix}1 & 8 & 2 & 7\\ 4 & 5 & 3 & 6\end{matrix}\right\}$

No. 27.

1	16	4	13
7	10	6	11
14	3	15	2
12	5	9	8

Second Group.

13th Arr.

$$\left\{\begin{matrix} 1 & 7 & 2 & 8 \\ 6 & 4 & 5 & 3 \end{matrix}\right\}$$

No. 28.

1	14	4	15
11	8	10	5
16	3	13	2
6	9	7	12

14th Arr.

$$\left\{\begin{matrix} 1 & 7 & 2 & 8 \\ 4 & 6 & 3 & 5 \end{matrix}\right\}$$

No. 29.

1	14	4	15
7	12	6	9
16	3	13	2
10	5	11	8

15th Arr.

$$\left\{\begin{matrix} 1 & 6 & 3 & 8 \\ 7 & 4 & 5 & 2 \end{matrix}\right\}$$

No. 30.

1	12	6	15
13	8	10	3
16	5	11	2
4	9	7	14

16th Arr.

$$\left\{\begin{matrix} 1 & 6 & 3 & 8 \\ 4 & 7 & 2 & 5 \end{matrix}\right\}$$

No. 31.

1	12	6	15
7	14	4	9
16	5	11	2
10	3	13	8

17th Arr.

$$\left\{\begin{matrix} 1 & 4 & 5 & 8 \\ 7 & 6 & 3 & 2 \end{matrix}\right\}$$

No. 32.

1	8	10	15
13	12	6	3
16	9	7	2
4	5	11	14

18th Arr.

$$\left\{\begin{matrix} 1 & 4 & 5 & 8 \\ 6 & 7 & 2 & 3 \end{matrix}\right\}$$

No. 33.

1	8	10	15
11	14	4	5
16	9	7	2
6	3	13	12

This makes 288 harmonic squares, of which one-third are perfectly harmonic the other 192 less perfectly so.

SERIES 3.—MAGIC SQUARES.

Yet another series, in which the squares add equally in every column and line and regularly in several other directions, but not diagonally, furnishes the means of producing thousands of ordinary magic squares.

In this series the trial arrangements add 18 in each line; the extremes add 18 and the means 18. The resulting squares add 34 vertically, horizontally, and other regular ways, but not diagonally. The numbered trial arrangements are here shown in order:

First Group.

19th.
1 8 5 4
7 3 2 6

20th.
1 8 5 4
6 2 3 7

21st.
1 7 6 4
8 3 2 5

22d.
1 7 6 4
5 3 2 8

23d.
1 7 6 4
5 2 3 8

24th.
1 6 7 4
8 2 3 5

25th.
1 6 7 4
5 2 3 8

26th.
1 6 7 4
5 3 2 8

27th.
1 5 8 4
7 3 2 6

28th.
1 5 8 4
7 2 3 6

29th.
1 5 8 4
6 2 3 7

30th.
1 5 8 4
6 3 2 7

Second Group.

31st.
1 8 3 6
7 5 2 4

32d.
1 8 3 6
4 2 5 7

33d.
1 7 4 6
8 5 2 3

34th.
1 7 4 6
3 5 2 8

35th.
1 7 4 6
3 2 5 8

36th.
1 4 7 6
8 2 5 3

37th.
1 4 7 6
3 2 5 8

38th.
1 4 7 6
3 5 2 8

39th.
1 3 8 6
7 5 2 4

40th.
1 3 8 6
7 2 5 4

41st.
1 3 8 6
4 2 5 7

42d.
1 3 8 6
4 5 2 7

Third Group.

43d.	44th.
1 8 2 7	1 8 2 7
6 5 3 4	4 3 5 6

45th.	46th.	47th.	48th.	49th.	50th.
1 6 4 7	1 6 4 7	1 6 4 7	1 4 6 7	1 4 6 7	1 4 6 7
8 5 3 2	2 5 3 8	2 3 5 8	8 3 5 2	2 3 5 8	2 5 3 8

51st.	52d.	53d.	54th.
1 2 8 7	1 2 8 7	1 2 8 7	1 2 8 7
6 5 3 4	6 3 5 4	4 3 5 6	4 5 3 6

Fourth Group.

55th.	56th.	57th.	58th.	59th.	60th.
1 7 2 8	1 7 2 8	1 7 2 8	1 7 2 8	1 7 2 8	1 7 2 8
6 5 4 3	4 3 6 5	5 6 3 4	5 3 6 4	3 5 4 6	3 4 5 6

61st.	62d.	63d.	64th.	65th.	66th.
1 6 3 8	1 6 3 8	1 6 3 8	1 6 3 8	1 6 3 8	1 6 3 8
7 5 4 2	4 2 7 5	5 7 2 4	5 2 7 4	2 5 4 7	2 4 5 7

67th.	68th.	69th.	70th.	71st.	72d.	73d.	74th.
1 5 4 8	1 5 4 8	1 5 4 8	1 5 4 8	1 5 4 8	1 5 4 8	1 5 4 8	1 5 4 8
7 6 3 2	7 3 6 2	6 7 2 3	6 2 7 3	3 7 2 6	3 2 7 6	2 6 3 7	2 3 6 7

75th.	76th.	77th.	78th.	79th.	80th.
1 4 5 8	1 4 5 8	1 4 5 8	1 4 5 8	1 4 5 8	1 4 5 8
7 3 6 2	6 2 7 3	3 7 2 6	3 2 7 6	2 6 3 7	2 3 6 7

81st.	82d.	83d.	84th.	85th.	86th.	87th.	88th.
1 3 6 8	1 3 6 8	1 3 6 8	1 3 6 8	1 3 6 8	1 3 6 8	1 3 6 8	1 3 6 8
7 5 4 2	7 4 5 2	5 7 2 4	5 2 7 4	4 7 2 5	4 2 7 5	2 5 4 7	2 4 5 7

89th.	90th.	91st.	92d.	93d.	94th.	95th.	96th.
1 2 7 8	1 2 7 8	1 2 7 8	1 2 7 8	1 2 7 8	1 2 7 8	1 2 7 8	1 2 7 8
6 5 4 3	6 4 5 3	5 6 3 4	5 3 6 4	4 6 3 5	4 3 6 5	3 5 4 6	3 4 5 6

To show what kind of squares this last series of arrangements will make, a few samples are here shown:

No. 34.
28th Arr.

1	10	16	7
13	4	6	11
8	15	9	2
12	5	3	14

No. 35.
49th Arr.

1	8	12	13
3	6	10	15
14	11	7	2
16	9	5	4

No. 36.
64th Arr.

1	12	6	15
9	4	14	7
16	5	11	2
8	13	3	10

No. 37.
96th Arr.

1	4	14	15
5	8	10	11
16	13	3	2
12	9	7	6

This makes 1,536 magic squares that may be constructed from that one scheme, 288 of which squares are harmonic.

INSTRUCTIONS.—To construct a primary square from one of the above trial arrangements:

1st. Draw a blank square of 16 places; and note the places for the odd numbers, as in the scheme.

2d. Apply the eight numbers of the trial arrangement selected, in their order, to the upper ends of the coupling lines, or conceive them to be so applied.

3d. Double each number of the trial arrangement, and place the products in spaces corresponding to both ends of the respective arrows, first subtracting 1 from the number falling in a space noted for an odd number.

SCHEME II (NO. 39 OF SCHEDULE).

Scheme II (figs. 33 and 34) is the same as scheme I turned over on its direct central diagonal as an axis. No. 1 still occupies the upper left-hand corner. The upper line in scheme I becomes the left-hand column in scheme II. All the columns of scheme I are converted into lines and the lines into columns. Fig. 34 is simply fig. 33 reversed, as fig. 26 was fig. 25 inverted, and as in that scheme seven of the derived squares will be like 33; eight of them like fig. 34. The coupling arrows in this scheme will be numbered at their left-hand end.

The first six arrangements, corresponding to the six harmonic arrangements of scheme I, are here shown with their corresponding squares:

1st Arrangement.
No. 38.

1 8
7 2
6 3
4 5

1	15	8	10
14	4	11	5
12	6	13	3
7	9	2	16

2d Arrangement.
No. 39.

1 8
7 2
4 5
6 3

1	15	12	6
14	4	7	9
8	10	13	3
11	5	2	16

3d Arrangement.
No. 40.

1 8
6 3
7 2
4 5

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

4th Arrangement.
No. 41.

1 8
6 3
4 5
7 2

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

5th Arrangement.
No. 42.

1 8
4 5
7 2
6 3

1	15	12	6
8	10	13	3
14	4	7	9
11	5	2	16

6th Arrangement.
No. 43.

1 8
4 5
6 3
7 2

1	15	14	4
8	10	11	5
12	6	7	9
13	3	2	16

All the other arrangements of scheme I, up to the 96th, are equally applicable to this scheme; and from each arrangement 16 squares can be constructed with equal facility with those of scheme I.

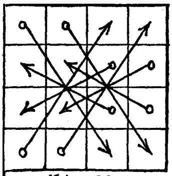


Fig. 33.

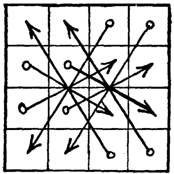


Fig. 34.

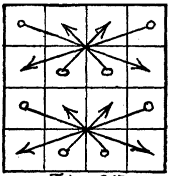


Fig. 35.

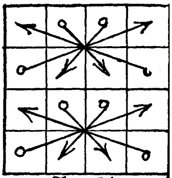


Fig. 36.

SCHEME III (No. 45 OF SCHEDULE).

This scheme (figs. 35 and 36) is like scheme I, except that the medial lines are transposed. In the trial arrangements the lines apply to the first and third (or upper alternate) lines of the square, instead of the first and second, as in scheme I. The upper line of the arrangements answers to the upper half of the square; the lower line to the lower half. All the harmonies and characteristics spoken of scheme I are equally true of this.

Samples of the harmonic squares yielded by this scheme are here shown:

1st Arr.

{ 1 7 6 4 }
{ 8 2 3 5 }

No. 44.

1	14	12	7
8	11	13	2
15	4	6	9
10	5	3	16

6th Arr.

{ 1 4 7 6 }
{ 8 5 2 3 }

No. 45.

1	8	14	11
12	13	7	2
15	10	4	5
6	3	9	16

Comparison of these squares may be made with similar arrangements in scheme I.

SCHEME IV (NO. 26 OF SCHEDULE).

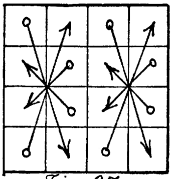


Fig. 37.

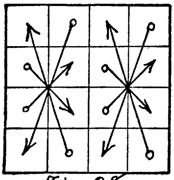


Fig. 38

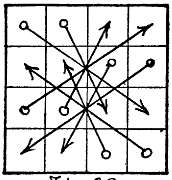


Fig. 39.

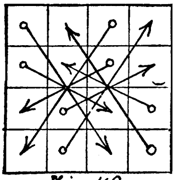


Fig. 40.

This scheme (figs. 37 and 38) is the same as scheme III turned over on its direct central diagonal, and as scheme II, with the central columns transposed.

The coupling arrows in this, as in scheme II, are to be numbered at their left-hand end. The vertical columns of the trial arrangements apply to the respective halves of the square.

One or two squares are given by way of comparison with those of scheme II.

2d Arrangement.
No. 46.

1 8
7 2
4 5
6 3

1	12	15	6
14	7	4	9
8	13	10	3
11	2	5	16

4th Arrangement.
No. 47.

1 8
6 3
4 5
7 2

1	14	15	4
12	7	6	9
8	11	10	5
13	2	3	16

All the arrangements shown for scheme I are equally applicable to the other three. This makes for the four schemes 384 arrangements of 16 squares each; or in all so far 6,144 squares, no two alike, of which 1,152 are perfectly harmonic, 384 of them to the highest degree.

OTHER SCHEMES.

But the foregoing schemes are not the only ones that can be made, though they are more harmonic than others. I will now introduce several other schemes and show at least one square made according to each by the first arrangement. It should be said here that the same arrangements will not answer for all alike; but each set of schemes has its own peculiarities and requires its own series of arrangements.

No. 48.

{ 1 7 6 4 }
{ 8 2 3 5 }

1	13	12	8
16	4	5	9
7	11	14	2
10	6	3	15

Scheme V (fig. 39) is a variation of scheme I. The variation consists in reversing the couplets of the second and fourth quaterniads, which in the scheme stand parallel with their complementary or opposite couplets. (Square No. 48.)

Scheme VI (fig. 40) differs from scheme II exactly as scheme V differs from scheme I and has the same characteristics. (See square No. 49.)

No. 49.

1 8
7 2
6 3
4 5

1	16	7	10
13	4	11	6
12	5	14	3
8	9	2	15

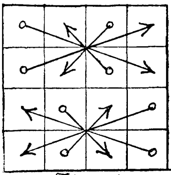


Fig. 41.

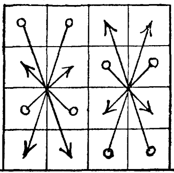


Fig. 42.

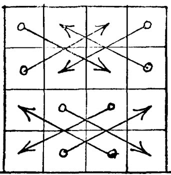


Fig. 43.

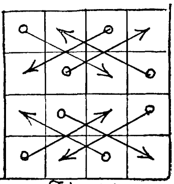


Fig. 44.

No. 50.

1	14	11	8
7	12	13	2
16	3	6	9
10	5	4	15

1 7 6 4
8 2 3 5

Scheme VII (fig. 41) is a variation of scheme III, and differs from it as scheme V differs from scheme I.

Scheme VIII (fig. 42) differs from scheme IV, as scheme V differs from scheme I. (See square No. 51.)

1 8
7 2
6 3
4 5

No. 51.

1	7	16	10
14	12	3	5
11	13	6	4
8	2	9	15

No. 52.

1	14	12	7
11	8	2	13
6	9	15	4
16	3	5	10

1 7 6 4
8 2 3 5

In scheme IX (fig. 43) the lines of the trial arrangement govern the outer opposite lines of the squares, instead of alternate or adjacent lines, as in former schemes. That is to say, the lines of the trial arrangement as they stand are to be applied to the first and fourth lines; their doubles, then, will fall in the outer lines of the square to be formed, and the mates in the inner lines.

In scheme X (fig. 44), also, the trial arrangement governs the outer opposite lines. (See square No. 53.)

1 7 6 4
8 2 3 5

No. 53.

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

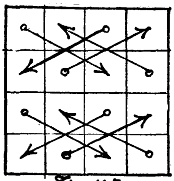


Fig. 45.

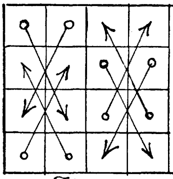


Fig. 46.

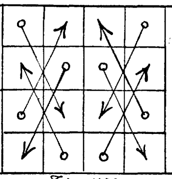


Fig. 47.

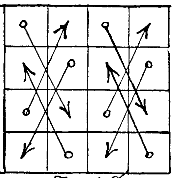


Fig. 48.

No. 54.

1	14	11	8
12	7	2	13
5	10	15	4
16	3	6	9

1 7 6 4
8 2 3 5

SCHEME XI—KNIGHT'S TOUR.—This scheme (fig. 45) is not harmonic; in a square of four it gives ordinary magic squares. Its defect is that it throws four even numbers into one diagonal and none into the other, thereby increasing that diagonal by 2 and decreasing the other by the same amount. But in a square of eight it does very well; because opposite quarters can be easily fitted to balance. The scheme differs from scheme X in reversing or inverting (technically condiverting) the lower half.

Scheme XII (fig. 46) is the same as scheme IX turned over on its direct central diagonal. The two columns of the trial arrangement govern the outer opposite columns of the squares.

1 8
7 2
6 3
4 5

No. 55.

1	11	6	16
14	8	9	3
12	2	15	5
7	13	4	10

No. 56.

1 8
7 2
6 3
4 5

1	12	6	15
14	7	9	4
11	2	16	5
8	13	3	10

Scheme XIII (fig. 47) is like scheme X turned over on its direct diagonal. It is governed the same as scheme X.

Scheme XIV, Knight's tour (fig. 48), can be used separately or in combination with scheme XI.

No. 57.

1 6
5 2
8 3
4 7

1	16	11	6
10	7	4	13
15	2	5	12
8	9	14	3

The regular transpositions allowed in the squares of all other schemes are not applicable to this or to scheme XI. These two schemes are necessarily governed by rules of their own.

About 40 other schemes may be constructed, all more or less defective, though many squares may be made from them, sometimes remedying the defect by transposing two pairs of contiguous couplets, or by other means. But schemes that are but poorly adapted to squares of four are well adapted to squares of eight; so that perfect squares of 64 places are much easier made than squares of 16 places, and can be made in much greater variety.

Here are two more schemes. The first is that used in laying 16-page forms in ordinary book printing.

Scheme XV.

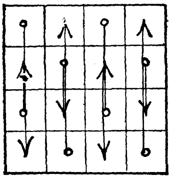


Fig. 49.

And the one arrangement necessary for proper folding is:

{ 1 8 7 2 }
{ 4 5 6 3 }

Which gives the primary square:

No. 58.
Primary.

1	16	13	4
8	9	12	5
7	10	11	6
2	15	14	3

This primary square shows the relative positions of the pages in the printed sheet before folding.

The pages are laid in *inverted* order on the imposing stone. They may also, for various reasons, be laid in three other positions, as shown in the four following transpositions:

No. 1.
Regular Form.
(Inverted.)

2 7	15 10	14 11	3 6
8 1	9 16	12 13	5 4

Printing first four pages outside.

No. 2.
Primary square indiverted.

14 11	3 6	2 7	15 10
12 13	5 4	8 1	9 16

Throwing last four pages outside.

No. 3.
Primary square converted.

8 1	9 16	12 13	5 4
2 7	15 10	14 11	3 6

Placing last four pages in center.

No. 4.
Primary square condiverted.

12 13	5 4	8 1	9 16
14 11	3 6	2 7	15 10

Making-up first four pages in center.

In the case of music books, where the pages are longest from front to back, the forms are made up by a different scheme. This is shown in Scheme XVI.

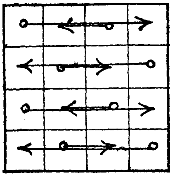


Fig. 60.

The arrangement of the couplets is:

1 8
4 5
3 6
2 7

And the primary square is:

No. 58.
Primary Square.

1	16	15	2
8	9	10	7
5	12	11	6
4	13	14	3

No. 5.
Regular Form.

4 5	13 12	14 11	3 6
8 1	9 16	10 15	7 2

With first four pages outside.

No. 6.
Indiverted Form.

14 11	3 6	4 5	13 12
10 15	7 2	8 1	9 16

With last four pages outside.

No. 7.
Converted Form.

8	9	10	7
1	16	15	2
4	13	14	3
5	12	11	6

With last four pages in center.

No. 8.
Condiverted Form.

10	7	8	9
15	2	1	16
14	3	4	13
11	6	5	12

With first four pages in center.

In all the foregoing forms, the outer columnar pairs are equal; the inner columnar pairs are equal; each line adds 34; each *pair* of pages adds 17; each quarter adds 34; in short, each pair of pages with any other pair adds 34. In all the transpositions under either scheme, the pages of any one line are the same in all; the pages of any one column are the same in all; the pages of any one quarter are the same in all the forms.

SCHEDULE OF SCHEMES.

In this schedule, the "first couplet" will be understood to occupy the upper left-hand corner place and another place to be indicated in the schedule. The "adjacent couplet" will occupy the adjacent space below, and another place which is indicated in the terms of the "first couplet." The "adjacent pair" will signify the other half of the upper left-hand quarter (a columnar pair) and the pair that is coupled with them. The "adjacent quarter" is the upper right-hand quarter. The "opposite half" is in this case the lower half.

	First couplet.	Adjacent couplet below.	Adjacent pair (beside).	Adjacent quarter beside.	Opposite half (below).
1	Same col. adj. line,	....	parallel,	inverted,	similar.
2	" "	....	"	"	reversed.
3	" "	....	inverted,	similar,	similar.
4	" "	....	"	"	reversed.
5	" "	....	"	contrary,	similar.
6	Same col. alt. line,	concurrent,	"	similar,	....
7	" "	"	"	contrary,	....
8	" "	recurrent,	"	similar,	....
9	" "	"	similar,	dissim.,	....
10	Same col. opp. line,	concurrent,	"	similar,	....
11	" "	"	"	contrary,	....
12	" "	recurrent,	"	inverted,	....
13	" "	"	inverted,	similar,	(Sch. XV.)
14	Adj. col. same line,	parallel,	....	"	reverse.
15	" "	"	....	reverse,	dissim.
16	" "	reversed,	....	similar,	similar.
17	" "	"	....	"	reverse.
18	" "	"	....	reversed,	similar.
19	Adj. col. adj. line,	transverse,	....	similar,	reverse.
20	" "	"	....	reverse,	dissim.
21	" "	reverse,	....	inverted,	similar.
22	" "	"	....	"	reverse.
23	Adj. col. alt. line,	reversed,	similar,	dissim.,	(Sch. XII.)
24	" "	"	dissim.,	contrary,	(Sch. XIII.)
25	" "	"	"	similar,	(Sch. XIV.)
26	Adj. col. opp. line,	"	"	(Sch. IV),	similar.
27	Adj. col. opp. line,	reversed,	similar,	(Sch. VIII),	inverted.
28	Alt. col. same line,	parallel,	reversed,	....	dissim.
29	" "	reversed,	dissim.,	....	similar.
30	" "	"	similar,	....	"
31	" "	"	"	....	dissim.
32	Alt. col. adj. line,	concurrent,	reversed,	(Sch. IX),	"
33	" "	recurrent,	dissim.,	(Sch. X),	inverted.
34	" "	"	"	(Sch. XI),	similar.
35	Alt. col. alt. line,	reversed,	"	dissim.,	....
36	" "	"	"	similar,	....
37	" "	"	similar,	dissim.,	....
38	" "	parallel,	reversed,	similar,	....
39	Alt. col. opp. line,	recurrent,	similar,	dissim.,	(Sch. II).

	First couplet.	Adjacent couplet below.	Adjacent pair (beside).	Adjacent quarter beside.	Opposite half (below).
40	Alt. col. opp. line,	concurrent,	reversed,	similar,	(Sch. VI).
41	Opp. col. same line,	parallel,	"	....	reversed.
42	" "	reversed,	similar,	....	dissim.
43	" "	"	dissim.,	(Sch. XVI),	similar.
44	" "	"	"	....	reversed.
45	Opp. col. adj. line,	recurrent,	"	(Sch. III),	"
46	" "	concurrent,	reversed,	(Sch. VII),	"
47	Opp. col. alt. line,	parallel,	"	dissim.,	(Sch. I)
48	" "	reversed,	similar,	"	(Sch. V).
49	Opp. col. opp. line,	concurrent,	reversed,	"	....
50	" "	recurrent,	similar,	"	....
51	" "	"	dissim.,	similar,	....
52	" "	"	"	dissim.,	....

The above schedule does not include schemes that do not fulfil the requirements—to provide for "two odd numbers, and no more, in each column, line, and diagonal." Even the schemes in the above schedule differ very much as to their capacity. Those selected and shown by diagram (Schemes I to IV et seq.) are the most perfect.

SPECIAL SUMS AND VARYING SERIES.

Suppose it be desired to construct a square that will add a certain number in each column or line, it is only necessary to *take as extremes of the series two numbers whose sum equals half of the number desired;* then select the other fourteen numbers of the series between those two.

Nor is it necessary that the several terms of the series shall be equidistant; it is only necessary that the common differences shall be harmonic, rhythmic, or concordant. For instance, the series 1, 3, 6, 8, 12, 14, 17, 19, 23, 25, 28, 30, 34, 36, 39, 41 is harmonic; because the differences rise and fall with a certain rhythm that fully satisfies the requirement. In other words, the differences 2, 3, 4, recur regularly.

The following examples are given to show some of the possibilities of these harmonic squares. These squares are made according to square No. 1, 1st arrangement, Scheme I:

No. 59.
Common differences 1, 1, 1, 2.

1	15	13	7
16	4	6	10
8	12	14	2
11	5	3	17

Sums = 36.

No. 60.
Common differences 1, 1, 2, 1.

1	16	13	8
17	4	7	10
9	12	15	2
11	6	3	18

Sums = 38.

No. 61.
Common differences 1, 1, 2, 2.

1	17	14	8
18	4	7	11
9	13	16	2
12	6	3	19

Sums = 40.

No. 62.
Common differences 1, 1, 2, 3.

1	18	15	8
19	4	7	12
9	14	17	2
13	6	3	20

Sums = 42.

Or suppose, for instance, it be desired to make the sums equal 100, it can be done in many different ways. The following examples from varying series will show some of the modes of accomplishing this result:

No. 63.
Common differences 3, 3, 4, 4.

1	43	36	20
46	10	17	27
23	33	40	4
30	14	7	49

Sums = 100.

No. 64.
Common differences 1, 4, 8, 8.

1	44	35	20
48	7	16	29
21	34	43	2
30	15	6	49

Sums = 100.

No. 65.
Common differences 2, 4, 5, 6.

1	43	36	20
47	9	16	28
22	34	41	3
30	14	7	49

Sums = 100.

No. 66.
Common differences 2, 3, 6, 8.

1	44	36	19
47	8	16	29
21	34	42	3
31	14	6	49

Sums = 100.

All of the above squares are harmonic, and have all the characteristics of the squares of Scheme I, notwithstanding their varying differences.

There are four grades of differences in all these squares, namely:

1st. The differences between adjacent numbers, as between the first and second, the third and fourth, etc. There are eight of these.

2d. The differences between adjacent pairs, thus coming in the middle of each quaterniad. There are four of these.

3d. The differences between adjacent quaterniads. There are two of these.

4th. The difference between the two halves of a series. This is but one.

The least difference that can be used, without repetition of a number, is 1. Any excess of difference between the extremes of a series more than 1 for each term may be used entirely in the 4th grade, or may be distributed among all the grades. When there is an excess of 1 it must be used in the 4th grade, as in the square "Sums equal 36" above. An excess of 2 or more may be used in the 3d or 4th grade, or both, as in the three squares "sums equal 38, 40, and 42" above. An excess of 4 or more may be used in either or all of the last three grades. An excess of 8 or more may be distributed anywhere, as in the squares "sums equal 100" above. Excess of difference must be distributed according to the following

Rule.—So distribute the excess that the differences in any one grade will be exactly the same throughout the varying series.

This can easily be done thus:

A difference of 1 in the first grade equals.........	8
A difference of 1 in the second grade equals......	4
A difference of 1 in the third grade equals........	2
A difference of 1 in the fourth grade equals.......	1
Total	15

A difference of 1 in the first grade is equal to 2 in the second, 4 in the third, and 8 in the fourth, as to results. Thus may any number be easily distributed and series so arranged that squares may be made to sum up any even number.

ODD SUMS.

Should it be desired to make a square add an odd number, it may be accomplished without fractions by making an extra difference of 1 in the last position of third grade, thus:

No. 67.

1	12	15	7
16	6	4	9
8	14	11	2
10	3	5	17

Sums = 35.

No. 68.

1	13	16	7
17	6	4	10
8	15	12	2
11	3	5	18

Sums = 37.

In which an extra difference of 1 is placed between the third and fourth sets. These squares, which add 35 and 37, respectively, are arranged according to the third arrangement of scheme I, as shown in square No. 18, that arrangement being best adapted to this purpose. But such squares, though they answer every requirement of a "magic square," are nor harmonic.

SUMS LESS THAN 34 AND NEGATIVE SUMS.

In forming these sums, the rule is just the same as for forming special sums, namely, to take for the extremes of the series two numbers whose sum equals half of the number desired; but in this case it presupposes the necessity of taking for the lower numbers of the series minus or negative numbers. A few squares, to show the manner of their formation, will be sufficient to give a good insight into this:

No. 69.

0	11	13	6
14	5	3	8
7	12	10	1
9	2	4	15

Sums = 30.

No. 70.

-4	8	10	3
11	2	-1	5
4	9	7	-3
6	-2	1	12

Sums = 17.

No. 71.

-4	8	10	2
11	1	-1	5
3	9	7	-3
6	-2	0	12

Sums = 16.

No. 72.

-8	3	5	-2
6	-3	-5	0
-1	4	2	-7
1	-6	-4	7

Sums = —2.

These squares are perfectly harmonic.

NUMBERS OF SERIES.

The simple arithmetical series 1 to 16, adding in the squares 34, is only one. That is to say: there is only one series of positive integers that can be found, without repetition, that will, when the numbers are arranged in a square, add 34 in each line, column, diagonal, etc. The same is true of the series adding 36; there is only one series. There are three series adding 38, and three adding 40. Thus, the series adding 38 are:

1.— 1 2 3 4 5 6 7 8 11 12 13 14 15 16 17 18
2.— 1 2 3 4 6 7 8 9 10 11 12 13 15 16 17 18
3.— 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

The number of series adding in square 42 and 44 is 7 of each; the number of series adding 46 and 48 is 13 of each; the number adding 50 and 52 is 23 for each; adding 54 and 56 is 37; adding 58 and 60 is 57; the number of series that will each separately add in square 62 or 64 is 83. Thus the number of series that will produce a given number constantly rises in an ever-increasing ratio; and when we come to series adding in square 100, we find there are no less than 895 series, and of series adding 102, there are 1,085.

The ratio of increase is not irregular, but is peculiar. The mode of forming the series is similar to that of forming any figurate series, except that each successive sum, until the series of increase is reached, is duplicated or written twice. This is shown by braces, thus:

	1				1				1				1			
	1		1		2		2		3		3		4		4	
	1	1	2	2	4	4	6	6	9	9	12	12	16	16	20	20
Increase	1	2	4	6	10	14	20	26	35	44	56	68	84	100	120	etc.
Series complete	1	3	7	13	23	37	57	83	118	162	218	286	370	470	590	etc.
Sums	34	38	42	46	50	54	58	62	66	70	74	78	82	86	90	

POSSIBILITIES.

It has been shown that by the different arrangements and transpositions more than 6,000 squares (6,144) may be constructed from a single series (1 to 16) according to the first four schemes; and it is presumed that at a low estimate at least four times as many more may be made from the remaining 48 schemes that are possible as shown in the schedule, thus making upward of 30,000 squares, no two alike, adding 34 in 10 or more different ways, that may be built up from that one series.

Now, under these considerations (and they are capable of demonstration), it will be evident that in case of squares adding 38, which may be done by three different series, there may be three times as many, or 90,000 made; as to squares adding 42 or 44, there may be seven times as many, or over 200,000 made; and when it comes to squares adding 100, which can be done by 895 different series, the number of squares that may be constructed, all different, is 895 times 30,000, or say 27,000,000, and of squares adding 102 no less than 33,000,000, no two alike, all magic, and at least one-fourth of them, or say 8,000,000, perfectly harmonic. Such numbers are beyond easy comprehension, and almost beyond belief. By way of better conceiving the import of such immense numbers, it may be stated that if a person were to construct at the rate of say 25 16-space squares an hour (and that would be found reasonably rapid), and work four hours a day for 270 days in the year, and not get sick or tired, and live long enough, it would take him just 1,000 years to make the 27,000,000 different squares, "each adding 100 in 10 or more different ways." Can human penetration farther go? Ah, yes, immeasurably farther.

Square of Six.

The square of six is produced by multiplication, in various ways, of the square of three and the square of two. For instance: the thirty-six places may be divided into four quarters, and each quarter arranged according to the square of three; or the square may be divided into nine sections, and each section filled according to the square of two.

I here produce several as samples:

No. 1.

23	3	29	24	2	30
28	17	12	25	18	11
7	35	14	8	34	13
22	4	31	21	1	32
26	19	10	27	20	9
5	33	15	6	36	16

No. 2.

23	24	3	2	29	30
22	21	4	1	31	32
28	25	17	18	12	11
26	27	19	20	10	9
7	8	35	34	14	13
5	6	33	36	15	16

No. 3.

24	33	19	10	8	17
15	6	28	1	26	35
34	7	5	14	30	21
16	25	23	32	12	3
20	29	27	18	13	4
2	11	9	36	22	31

These squares are all laid like the scheme in fig. 2, Square of Three.

So far as the arrangement of the four numbers in each section, or in the respective places in the four quarters, is concerned, it is necessarily various; but since there are twenty-four methods of laying these numbers, as shown in Square of Two, and only nine of them can possibly be used, it allows a very wide latitude in the selection of methods.

The first step necessary in producing such a square is to arrange the numbers 1, 2, 3 and 4 into sections in such a way that they will add equally in each line, column, diagonal, and such other directions as may be chosen, thus:

A.

3	4	3	2	1	2
2	1	4	1	3	4
4	1	1	2	4	3
2	3	3	4	2	1
3	4	3	2	2	1
1	2	1	4	3	4

B.

1	3	4	2	2	3
2	4	3	1	1	4
3	4	1	3	3	1
2	1	2	4	2	4
3	2	4	2	3	1
4	1	1	3	4	2

C.

2	1	3	2	3	4
3	4	1	4	1	2
4	3	1	2	4	1
2	1	3	4	2	3
1	2	3	2	3	4
3	4	4	1	2	1

Etc., in many, many ways.

Then next, according to one of the schemes in the Square of Three, add to all the numbers in each of the sections the respective numbers 0, 4, 8, 12, 16, 20, 24, 28, and 32.

Or, divide the thirty-six numbers into four sets of nine each and take one from each set for each of the respective positions. This will be equivalent to adding to all the numbers of each section the amounts 0, 1, 2, 3, 4, 5, 6, 7, and 8, and to the several numbers in each section the amounts 0, 9, 18, and 27. Various other methods may be pursued.

In the above squares, Nos. 1, 2, and 3, it will be seen that not only does each line, column, and diagonal add 111, but that any four numbers taken regularly and at equal distances from the center add 74, also that the transverse semi-diagonals in opposite quarters add 111; also transverse tertio-diagonals add 111.

These squares bear transposing after a manner similar to the transpositions in the square of four. It is possible to transpose so as to bring any required number into the upper left-hand corner, and still preserve the harmony of the square.

A few examples are given of transpositions from square No. 3:

No. 4.

1	24	6	35	26	19
28	33	15	17	8	10
23	7	34	21	12	14
32	16	25	3	30	5
9	20	29	4	13	36
18	11	2	31	22	27

No. 5.

2	11	18	27	22	31
29	20	9	36	13	4
25	16	32	5	30	3
32	7	23	14	12	21
15	33	28	10	8	17
6	24	1	19	26	35

No. 6.

3	30	5	32	16	25
4	13	36	9	20	29
31	22	27	18	11	2
35	26	19	1	24	6
17	8	10	28	33	15
21	12	14	23	7	34

No. 7.

4	13	36	9	20	29
31	22	27	18	11	2
21	12	14	23	7	34
3	30	5	32	16	25
35	26	19	1	24	6
17	8	10	28	33	15

No. 8.

5	30	3	25	16	32
36	13	4	29	20	9
27	22	31	2	11	18
19	26	35	6	24	1
10	8	17	15	33	28
14	12	21	34	7	23

No. 9.

6	24	9	19	26	35
15	33	28	10	8	17
34	7	23	14	12	21
25	16	32	5	30	3
29	20	9	36	13	4
2	11	18	27	22	31

Any number in this square except 7, 9, 10, 12, 16, 28, 30, and 36, may be brought into the upper left-hand corner. These eight numbers immediately surround the central square of four numbers in square No. 9. These numbers, too, may be put into the same corner; but in that case the sums of the diagonal lines will not be 111.

Squares like the above may be produced almost without limit. They are, however, not harmonic. In explanation of this it may be said that a perfectly harmonic square of 6 places on a side is impossible; since, in every attempt at it there will always be found left four numbers that can only be put in according to the square of two, and since this adds equally only twice, while six additions are needed, a choice must be made.

I now present a square on the old-fashioned plan—that of transposing numbers from their regular order vertically and horizontally. There is no special merit in this; and I only present it to show that, in addition to the lines, columns, and diagonals adding regularly 111, the six numbers on either side of the center, and some curious basket-shaped figures (to be pointed out) near the center also add the same. For instance, the following numbers add together 111: 16, 23, 21, 27, 14, 10; 15, 10, 22, 20, 27, 17; 22, 14, 15, 9, 23, 28; 21, 28, 16, 17, 9, 20; 21, 13, 8, 4, 30, 35; 28, 9, 23, 15, 16, 20; 14, 21, 22, 17, 10, 27; 23, 14, 28, 15, 21, 10; and, finally, 9, 16, 22, 27, 20, 17. Almost any number will add one-third of 111 [37] with the number diametrically opposite; and any four numbers at the corners of a rectangle, rhomb, rhomboid, or square, having for

No. 10.

1	5	33	34	2	6
12	26	28	9	29	7
19	23	15	16	20	18
13	14	21	22	17	24
30	8	10	27	11	25
31	35	4	3	22	36

its center the center of the square, will add 74, or two-thirds of 111. Many other squares similar to this may be constructed, and some of them exhibit singular features, but none of them are of special merit.

Enough has been shown now by which any one may construct all the squares of 36 places that he may wish, and place any numbers in any desired position.

Square of Eight.

These squares are laid according to any of the schemes for the square of four, but not forgetting the square of two. They are very easy, and hundreds of thousands of them may easily be produced. For, schemes that do not readily work in a square of 4 do well in a square of 8, and none of them fail.

There are three general methods of constructing a square of 8, namely:

1st. To divide the square into sixteen sections, and arrange the square as a quadruple square of sixteen places.

2d. To arrange it in four separate squares of sixteen places each, side by side.

3d. To place one square in the center, two dimidiated squares, either right-side out or inside out, on the four opposite sides, and one quartered square at the four corners.

There are many modes of arranging the series to fill the respective squares, two of which are here shown:

First Method.

Here are shown two squares constructed according to the first method. They are laid according to the third arrangement of scheme I (see square No. 33), Square of Four, and the diagonal (No. 3) method of the Square of Two in its four different aspects:

No. 1.

1	3	48	46	54	50	27	31
5	7	44	42	53	52	25	29
59	63	22	18	16	14	33	35
57	61	24	20	12	10	37	39
32	30	49	51	43	47	6	2
28	26	53	55	41	45	8	4
38	34	11	15	17	19	64	62
40	36	9	13	21	23	60	58

No. 2.

1	17	60	44	46	14	23	55
33	49	28	12	62	30	7	39
31	63	38	6	52	36	9	25
15	47	54	22	20	4	41	57
56	40	13	29	27	59	34	2
24	8	45	61	11	43	50	18
42	10	19	51	5	21	64	48
58	26	3	35	37	53	32	16

In No. 1 the sixty-four numbers are taken as two series, the odd numbers in one series, the even numbers in another; then four consecutive odd numbers are taken in one group to represent an odd number, four even ones to represent an even number, then filling as a quadruple square of 4.

In No. 2 the sixty-four numbers are divided into four series, and the first number of each series is taken to fill the first quadruple member of the square, the second number to fill the second member, and so on.

These two are not the only ways that numbers may be taken to fill the squares. The sixty-four numbers may be divided into series of 32, 16, 8, 4, 2 or 1 places; the series may be regular, alternate, or half complementary; they may be direct or half reversed; each group of four may be taken in regular order. either direct, reversed, or crosswise of the series, or one-half may be complementary to the other half.

Any of the schemes and arrangements in Square of Four is applicable to this square; and many of them are better adapted to this square than to the Square of Four.

The harmony of the square, aside from the scheme and arrangement, is owing largely to

(a) The preparation of the series;
(b) The mode of selection of the groups to fill the quadruple squares; and
(c) The mode of laying the several groups.

Plan.

In laying any of these squares, in order to have them perfectly harmonic, it will be necessary to observe several precautions:

1st. The several groups of four numbers must be so commenced in the different quadruple squares that no two in the same double line or double column shall begin in the same relative position, and not more than two in each diagonal.

2d. Increase in the first square may be made in either of three directions; but no two of the quadruple squares in any column or line should increase in the same direction, and no more than two in the diagonals.

This may be accomplished by commencing the different quadruple squares according to a prepared plan, as either of the following, the numbers indicating the relative square in each quadruple square:

A.

1	4	3	2
2	3	4	1
4	1	2	3
3	2	1	4

B.

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

C.

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

D.

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

And increase in each of the quadruple squares may be made thus:

right	up	down	left
left	down	up	right
up	right	left	down
down	left	right	up

These plans are offered merely as suggestions. It will readily be seen that many different plans and modes may be adopted, including four diagonal directions.

Second Method.

Here are presented the same squares as Nos. 1 and 2, constructed according to the second method:

No. 3.

1	48	54	27	3	46	50	31
59	22	16	33	63	18	14	35
32	49	43	6	30	51	47	2
38	11	17	64	34	15	19	62
5	44	56	25	7	42	52	29
57	24	12	37	61	20	10	39
28	53	41	8	26	55	45	4
40	9	21	60	36	13	23	58

No. 4.

1	60	46	23	17	44	14	55
31	38	52	9	63	6	36	25
56	13	27	34	40	29	59	2
42	19	5	64	10	51	21	48
33	28	62	7	49	12	30	39
15	54	20	41	47	22	4	57
24	45	11	50	8	61	43	18
58	3	37	32	26	35	53	16

These squares are sufficiently harmonic to add 130 in all the different ways indicated in Square of Four.

No. 3 is obtained from No. 1 by transposition; No. 4 from No. 2.

Third Method.

Here, now, are two squares written by the third method indicated. They are derived by transposition from Nos. 3 and 4, respectively:

No. 5.

1	48	3	46	50	31	54	27
59	22	63	18	14	35	16	33
5	44	7	42	52	29	56	25
57	24	63	20	10	39	12	37
28	53	26	55	45	4	41	8
40	9	36	13	23	58	21	60
32	49	30	51	47	2	43	6
38	11	34	15	19	62	17	64

No. 6.

1	60	17	44	14	55	46	23
31	38	63	6	36	25	52	9
33	28	49	12	30	39	62	7
15	44	47	22	4	57	20	41
24	45	8	61	43	18	11	50
58	3	26	35	53	16	37	32
56	13	40	29	59	2	27	34
42	19	10	51	21	48	5	64

These squares, and all other squares built upon some definite plan, are perfectly harmonic, both as wholes and as parts.

Knight's Tour Squares.

These squares are constructed according to Schemes XI and XIV of the Square of Four. They solve the problem of passing the knight by his peculiar leaps over the sixty-four squares of the chess-board without doubling or missing a square.

These squares are not harmonic. Of the thousands of different solutions of the problem very few are harmonic to any degree.

There are three methods of constructing squares according to these schemes in order to have them at all harmonic.

1st. Reversing the direction after passing once around with each series of sixteen numbers, thus making three reversals.

2d. Reversing after placing the first series, running twice around in the contrary direction, then reversing again at the end of the third series.

3d. Reversing once for all after placing the first two series.

Four squares are here shown, illustrating the methods of making squares of this kind:

No. 7.

19	64	47	2	49	32	15	34
46	3	18	63	16	35	50	31
61	20	1	48	29	52	33	14
4	45	62	17	36	13	30	51
21	60	5	44	53	28	37	12
6	43	24	57	40	9	54	27
59	22	41	8	25	56	11	38
42	7	58	23	10	39	26	55

One turn.

No. 8.

31	2	47	50	33	64	15	18
48	51	32	1	16	19	34	63
3	30	49	46	61	36	17	14
52	45	4	29	20	13	62	35
5	28	41	56	9	60	37	22
44	53	8	25	40	21	12	59
47	6	55	42	57	10	23	38
54	43	26	7	24	39	58	11

Two turns.

No. 9.

63	2	31	34	47	50	15	18
32	35	62	1	14	17	48	51
3	64	33	30	49	46	19	16
36	29	4	61	20	13	52	45
5	60	25	40	9	44	21	54
28	37	8	57	24	53	12	43
59	6	39	26	41	10	55	22
38	27	58	7	56	23	42	11

Three Turns.

No. 10.

5	44	19	38	3	54	21	28
18	37	4	43	20	27	2	53
45	6	39	26	55	10	29	22
36	17	58	7	42	23	52	1
59	46	25	40	9	56	11	30
16	35	8	57	24	41	64	51
47	60	33	14	49	62	31	12
34	15	48	61	32	13	50	63

Endless Chain.

These squares are very variable, but are not transposable except by inversion, reversion, reinversion, and rotiversion. The number 1 may be placed anywhere; but, if moved from its present position, the little harmony the squares here have is broken. The four squares above are all "endless chains."

U. S. GRANT HARMONIC SQUARE.

The following square, first published in the Topeka *Daily Capital* July 29, 1885, the day of General Grant's funeral, is shown here to illustrate one of the practical applications to which harmonic forms may be applied. It contains all the years of General Grant's life systematically arranged:

1864	1823	1883	1844	1840	1834	1855	**1885**
1874	1853	1857	1830	1879	1861	1832	1842
1829	1866	1838	1881	1827	1849	1876	1862
1847	1872	1836	*1859*	*1868*	1870	1851	1825
1882	1856	1837	*1839*	*1848*	1871	1835	1860
1845	1831	1858	1880	1826	1869	1841	1878
1865	1871	1846	1828	1877	1850	1854	1833
1822	1852	1873	1867	1863	1824	1884	**1843**

"The four corners represent eras or points of greatest prominence in his life. His first entry into life was in 1822, in the lower left-hand corner. The next great prominent event of his life was his graduation from the military academy and entry into the army in 1843, found in the lower right-hand corner. The third important era in his life, when he first came most prominently before the American people, was in 1864, when he was made lieutenant-general, and when he "proposed to fight it out on this line if it took all summer," which he did. This date is the most conspicuous, being in the upper left-hand corner. The final prominent number is the present year, the date of his triumphal departure, to be found in the upper right-hand corner. His life, considered as a harmonic square, is complete.

"Let it be observed that the difference between these most prominent dates is exactly twenty-one years, thus marking his life into three periods of equal duration, namely: The first, that of youth and study; the second, that of diligent labor; and the third, that of triumphs and rewards. The sum of these four numbers is 7,414.

"The four central numbers of the above square represent also four notable dates in his private or public life, less important than the other four, namely: 1839, his entry into West Point military academy; 1848, his marriage; 1859, his industrious laboring as a tanner at Galena, Ill., which fact figured so prominently in his first presidential campaign; and 1868, his election to the presidency of the United States. The sum of these four numbers is 7,414.

"Aside from the foregoing statements, the peculiar properties of the above square are such that each line, column, or diagonal, foots up 14,828. Any semi-diagonal or diagonal of four places, added to the opposite semi-diagonal, sums up 14,828. Any half line or half column, commencing at one end, added to any other half line or half column, makes 14,828. Any four in a square inclosed by heavy lines, added to any other similar square, is 14,828. Any four numbers whatever, taken at random anywhere, added to four taken similarly and diametrically opposite, will equal 14,828. The foregoing statements will afford several hundred thousand ways of footing up 14,828.

"Divide the entire square into four equal parts. Each quarter is harmonic by itself. Each line and each column in each quarter sums up 7,414. Each transverse or direct semi-diagonal of two places, added to an opposite transverse or direct semi-diagonal of the same quarter, adds 7,414.

"Divide the square into sixteen equal parts. The sum of the four numbers of each such part is 7,414. Any pair of numbers, added to a similar pair diametrically opposite, equals 7,414. Two numbers taken anywhere, added to the two numbers diametrically opposite, equal 7,414.

"Finally, any number, added to the number diametrically opposite, is equal to 3,707. This number is composed of the prime factors 11 and 337.

"It is believed that the footings indicated, together with many others not mentioned, can be obtained in more than a million different ways, or say as many as there were men in the federal army."

SECTION II. OTHER PLANE FORMS.

HEXAGONS.

I present a harmonic hexagon (fig. 51), which will, no doubt, win admiration for its beautiful symmetry and perfect harmony.

Harmonic Hexagon.

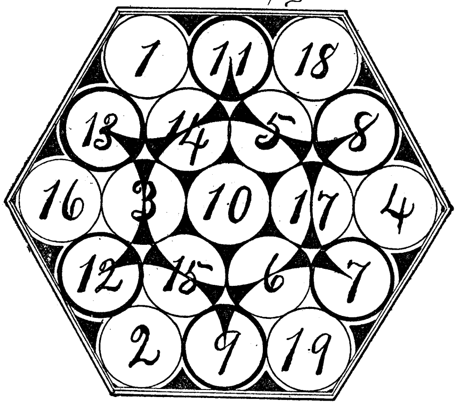


Fig. 51.

In this hexagon any full line of three places adds 30, whether along one side or through the center; any line of four places equals 40; of five places 50. The inner circle of six adds 60; the outer circle of 12 adds 120. The numbers in the middle of each of the sides, being at the apices of two large triangles (as 11, 12, 7), add 30 for each three. The same six numbers, each taken separately in connection with the two adjacent numbers of the inner circle, thus forming six small triangles, adds each 30, as 11, 14, 5. Any two opposite numbers add 20; any four, forming a parallelogram (rectangle, rhomb, or rhomboid) having its center at center of hexagon, add 40; the extremes of any outer line and the means of the opposite interior line, as 1, 18, 6, 15, add 40; any pair at one end of an outer line, added to the adjacent pair of an alternating outer line, as 1, 11, 16, 12, add 40. The three numbers on any side, added to either of six other possible pairs, add 50. Six numbers in any order, three on one side and three diametrically opposite, add 60; the three of any outer line and the three of an alternating outer line, as 1, 11, 18, 2, 12, 16, add 60. Any eight, four on either side, complementary to four opposite, add 80. Alternate outer lines sum up 90; adjacent pairs of alternate outer lines, with the transverse central row, as 1, 11, 16, 12, 2, 15, 10, 5, 18, add 90; any small triangle and two adjacent sides equal 90; same with two alternate sides equal 90; same with opposite and one adjacent side equal 90; two alternate small triangles and either of four sides equal 90. Any ten, half complementary to other half, equal 100; any twelve, ditto, equal 120; any fourteen, the same, 140; any sixteen 160.

Thus, in the above hexagon, we may find nine sums of 20 each; 23 sums of 30 (14 independent and the nine sums of 20 added to the central number); 57 sums of 40; 93 sums of 50; 147 sums of 60; the same of 70 (the same sums with central number added); 126 sums of 80; 82 sums of 90; 70 sums of 100; the same of 110; 60 sums of 120; the same of 130; 28 sums of 140; 42 sums of 150; 23 sums of 160; 9 sums of 170; one sum of 180 and one of 190. In all cases, whatever numbers are added, if they are added as indicated, will add 10 for each number taken, the same as if the numbers were all 10's.

TRANSPOSITIONS.

By inverting, reversing, etc., as in the squares, twelve apparently different harmonic hexagons may be obtained from this one; but which are in reality only different views of the same hexagon.

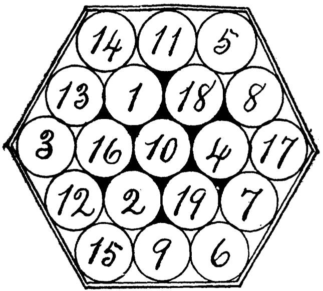


Fig. 52.

Again, by transposing the inner circle for the outer corners, number for number, an apparently new hexagon (fig. 52) is formed from this; but which a careful study will show is only one form of transposition; and which may also, by reversion, inversion, etc., be transformed into twelve other hexagons, making twenty-four in all. All these hexagons have the same harmonic properties as the first. Whatever is true of one is true of all.

Four of those hexagons, showing apparent differences, are here presented:

2 12 16	6 7 17	3 12 15	18 11 1
9 15 3 13	9 19 4 8	13 16 2 9	8 5 14 13
19 6 10 14 1	15 2 10 18 5	14 1 10 19 6	4 17 10 3 16
7 17 5 11	12 16 1 11	11 18 4 7	7 6 15 12
4 8 18	3 13 14	5 8 17	19 9 2

In figure 54 a second harmonic hexagon is presented, which has all the peculiarities ascribed to figure 51, and which is capable of the same number of transpositions, one of which is shown in fig. 55, appearing like an altogether different harmonic hexagon.

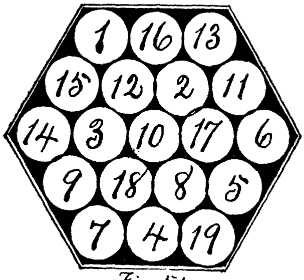


Fig. 54.

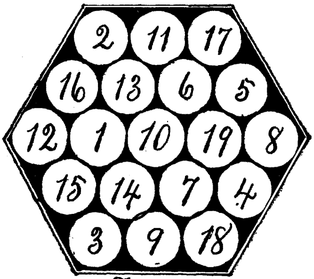


Fig. 55.

In fig. 56 a third harmonic hexagon is shown, having the same peculiarities and capable of the same transpositions as the other two, one of which is shown in fig. 57.

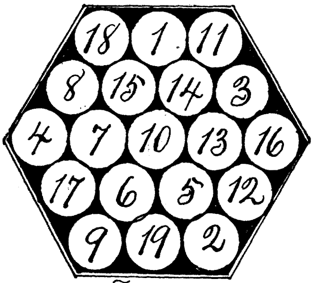


Fig. 56.

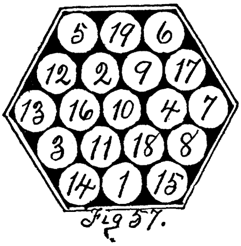


Fig. 57.

Figure 58 presents a fourth harmonic hexagon, with the same peculiarities and capable of the same transpositions as the others, one of which is shown in fig. 59.

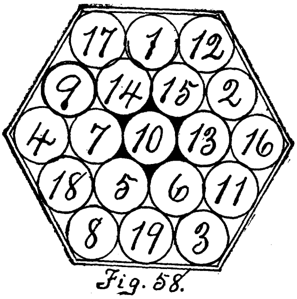


Fig. 58.

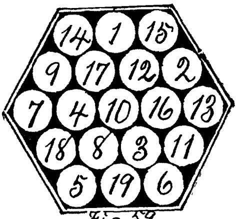


Fig. 59.

This makes ninety-six harmonic hexagons, all having the same peculiarities as the hexagon in fig. 51, and equally harmonic in every respect.

A harmonic hexagon of 37 places has not yet been constructed. A magic hexagon of 37 places is presented in fig. 60, each row of which, in any direction, adds as many times the central number as the number of places in the row. This is not harmonic.

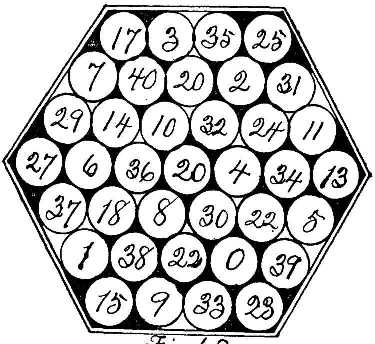


Fig. 60.

PENTAGONS.

A magic pentagon is presented in figure 61, in which the rows of numbers on either side add as many times the central number as the number of pentagons in the row. The five pentagons in any corner added to the middle pentagon at the opposite side, adds six times the central number.

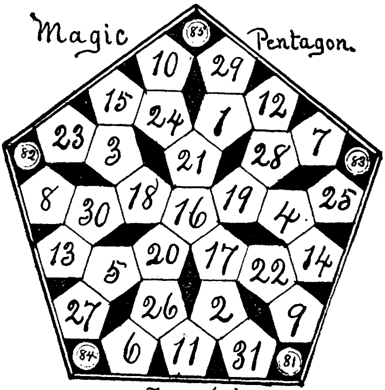


Fig. 61.

The harmony of arrangement of the numbers may readily be seen on inspection.

OCTAGONS.

A single octagon (a modified circle) is presented in figure 62, in which the alternate series is used instead of the numerical series. In this octagon

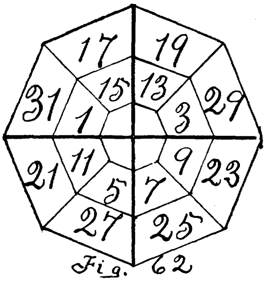


Fig. 62

the outer circle is exactly three times the diameter in any direction. Each of the quarters is equal to the diameter, and the sum of the inner circle is the same. There is nothing remarkable about this.

RHOMBS.

In figure 63 is presented a harmonic rhomb or diagonal square, in which

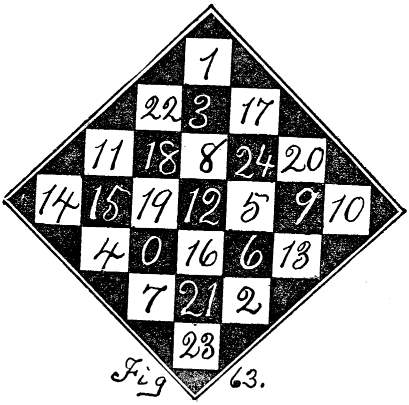


Fig 63.

the full lines in any direction are always as many times the central number as the number of places in the line. Black and white numbers may be added separately or together in the central lines.

STARS AND CIRCLES.

In figure 64 is presented a six pointed star, which is, so far, the most harmonic in every respect of all forms presented. Lines, triangles, and circles, whether taken at one side or through the middle, always add as many times the central number as the number of places taken. Everything that has been said of the harmonic hexagon is equally true of this.

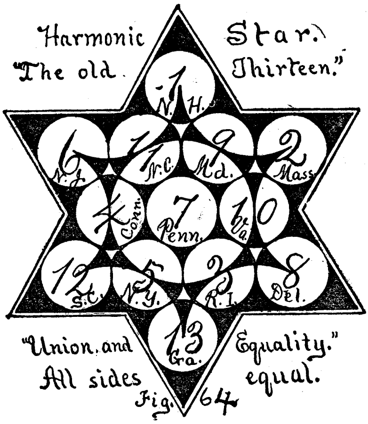


Fig. 64

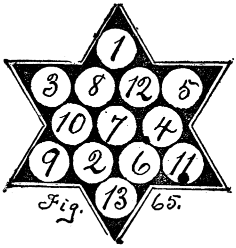
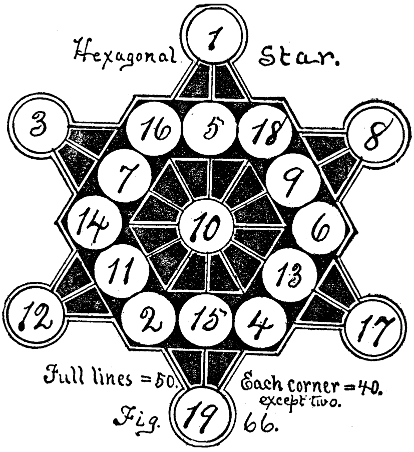


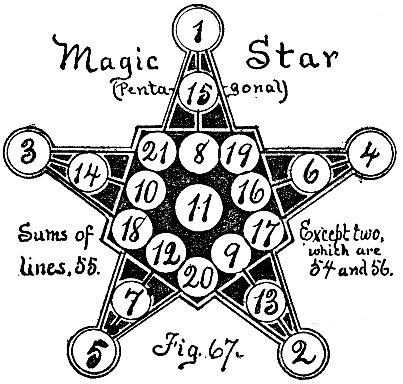
Fig. 65.

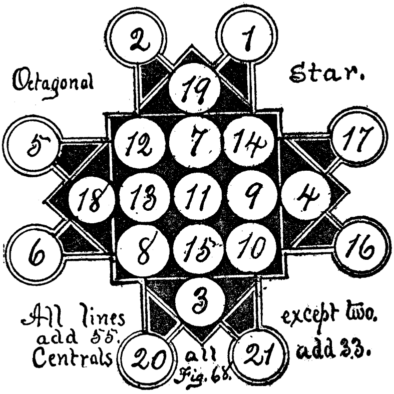
There are only thirteen numbers in it; but, without changing the position of the central number, which is the keystone upon which the entire star rests, it is capable of 24 transpositions the same as the harmonic hexagon. As a specimen of symmetry and perfect harmony, it excels all forms yet presented. One of the transpositions is shown in figure 65.

A new hexagonal star is presented in figure 66, in which each of the ten full lines of five places adds five times the central number, and each of the corners except two, with the adjacent three places, adds four times the central number. This star is not harmonic.



A magic pentagonal star is shown in figure 67, in which each of the full lines except two adds five times the central number.





In the octagonal star (figure 68) each of the full lines except two adds five times the central number. The square in the center adds equally in every direction. It is not harmonic, though many arrangements may be made.

In the small circle (fig. 69) the circumference equals three times the diameter in both inner and outer circles, adding diameters of circles separately.

In figure 70, which is the square of 4 in a new form, the diameter and the quarters are equal, as in the case of the octagon (fig. 62), which is a modification of this; and the circumference, owing to the series, is slightly less than three times the diameter. The series in figure 62 is adapted to making the circumference equal three times the diameter.

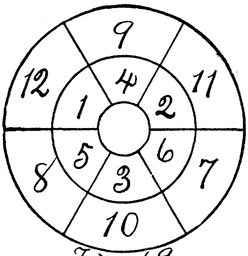


Fig. 69.

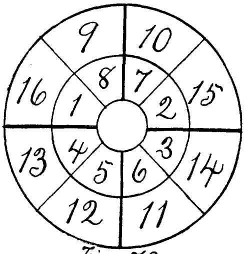


Fig. 70.

In figure 71, which is a further modification of the same circle, the circumference bears relation to the diameter as 3.1416 to 1. Each of the quarters add 1 equally with the diameter. The sum of the inner circle (.8584) is equal

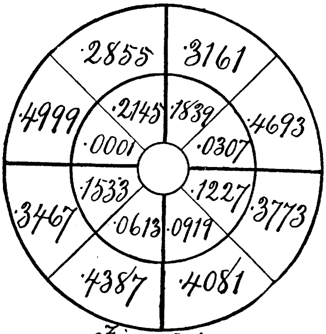


Fig. 71.
$D:C=1:\pi$

to the difference between 3.1416 and 4. This series is only one of many thousands that may be made to produce this result; and is presented because it happened to be the first one tried, and to show that series may be arranged in any form to produce any reasonable desired result. The series can just as well be extended to produce π (Pi) to any number of decimal places without the least variation.

In figure 72 the trigonal series is used instead of an arithmetical series; yet the relations of diameters to each other are not changed. Diameters alike equal 380.

Trigonal Series.

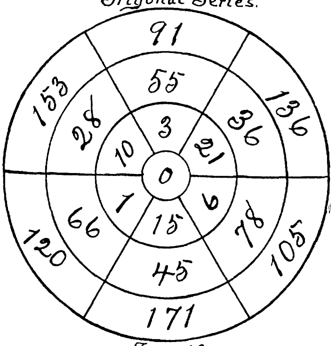


Fig. 72.

Square Series.

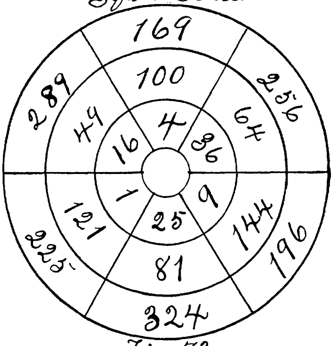


Fig. 73.

In figure 73 the square series is inserted with the same result. Diameters equal 703.

These series, trigonal and square, are placed in the form of Greek crosses, in which the columns and lines all add equally. (Figs. 74 and 75.)

Trigonal Series.

1	78	105	28
120	21	10	45
36	91	66	3
55	6	15	136

Fig. 74.

Square Series.

1	144	196	49
225	36	16	81
64	169	121	4
100	9	25	256

Fig. 75.

Besides columns and lines adding equally in the above crosses, the following pairs of equal sums are obtainable: Opposite quarters are equal; curves, having for their centers the centers of the crosses, or the middle points of the termini of the crosses, and consisting of the means of one column or line and the extremes of the adjacent one, are equal to a parellel curve in the opposite half of the square. For instance: 78 + 120 + 36 + 6 = 28 + 10 + 66 + 136. The corner numbers, outside the cross, are to be considered in calculating quarters and curves.

In figure 76 is presented a parallelogram transformed into a circle, in which all diameters, circles, and prolate cycloidal lines (indicated by dotted lines) add equally, as they would do if the numbers were arranged in a rectangle.

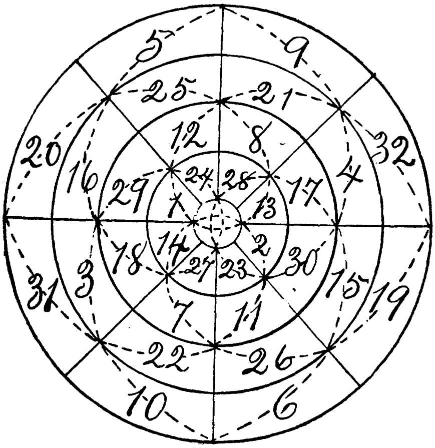


Fig. 76.

The above circles and squares are all perfectly transposable, without marring the harmony of arrangement of the numbers within the figures.

Other plane forms and all volumetric forms will be left to a future time. Many of them are already prepared; but there is much to do yet.

FORMULA.

To determine the sum of any line in any square, rectangle, star, or circle, multiply half the sum of the extremes of the series by the number of places in a line. Expressed algebraically, the formula would appear thus:

$$\frac{(l+f)}{2}p=s$$

In which l represents the last term of the series; f represents the first term; p represents the number of places in a line; and s stands for sum. This is true of any series, whatever the first term and whatever the common difference. In nearly all forms given heretofore, the first term of every series has been 1 and the common difference 1; yet whatever is said of those series is equally true of any other series similarly arranged.